Embedding a manifold in $\mathbb{R}^n$

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1: Embedding Lemma. *Suppose* $\mathbf{M}$ *is a compact* $\mathfrak{D}$*-dimensional manifold* (*with* $\mathfrak{D}$ *finite*). *Then there exists a positive integer* n *such that* $\mathbf{M}$ *can be embedded in* $\mathbb{R}^n$. *This means that there exists a homeomorphism*

$$f : \mathbf{M} \hookrightarrow \widehat{\mathbf{M}} \subset \mathbb{R}^n ,$$

where $\widehat{\mathbf{M}}$ *has the topology induced by* $\mathbb{R}^n$. ♢

Proof. It suffices to find a continuous injection $f : \mathbf{M} \hookrightarrow \mathbb{R}^n$. For then f is a continuous bijection from the compact space $\mathbf{M}$ to the Hausdorff space $\widehat{\mathbf{M}} := f(\mathbf{M})$ and consequently is a homeomorphism.

To construct f, cover $\mathbf{M}$ with some finite collection $\mathcal{C}$ of patches i.e,

$$\mathbf{M} \subset \bigcup_{P \in \mathcal{C}} P , \qquad \text{each patch equipped with an } \underline{\text{onto}} \text{ homeomorphism } h_P : P \to \mathbb{R}^{\mathfrak{D}} ,$$

Suppose, for each $P \in \mathcal{C}$, that we could find a continuous map $\varphi_P : \mathbf{M} \to \mathbb{R}^{\mathfrak{D}+1}$ with

$$[x, y \in P \text{ and } x \neq y] \implies \varphi_P(x) \neq \varphi_P(y) .$$

Then the cartesian product map,

$$f := \prod_{P \in \mathcal{C}} \varphi_P ,$$

would be a continuous injection (as desired) from $\mathbf{M}$ into $\mathbb{R}^n$, where $n := |\mathcal{C}| \cdot [\mathfrak{D}+1]$.

Now let $\mathbb{S}^{\mathfrak{D}}$ denote the unit $\mathfrak{D}$-sphere inside of $\mathbb{R}^{\mathfrak{D}+1}$. We know that $\mathbb{R}^{\mathfrak{D}}$ "extended by a point at infinity" is homeomorphic with $\mathbb{S}^{\mathfrak{D}}$. Let

$$\Phi : \mathbb{R}^{\mathfrak{D}} \sqcup \{\infty\} \hookrightarrow \mathbb{S}^{\mathfrak{D}}$$

be one such homeomorphism; say, stereographic projection.

To obtain φ_P, notice that the patch homeomorphism $h_P : P \to \mathbb{R}^{\mathfrak{D}}$ can be extended to a continuous map

$$\widehat{h}_P : \mathbf{M} \to \mathbb{R}^{\mathfrak{D}} \sqcup \{\infty\}$$

by $\widehat{h}_P(x) :=$ "∞" for all $x \in \mathbf{M} \smallsetminus P$. So we can create the desired map $\varphi_P : \mathbf{M} \to \mathbb{R}^{\mathfrak{D}+1}$ by setting $\varphi_P := \Phi \circ \widehat{h}_P$. ♦

Whoa! 04May2010: But why should the above $\widehat{h}_P$ be continuous? We certainly can force $\widehat{h}_P$ to be cts, by insisting (which we can, WLOG) that the given h_P is a homeomorphism *onto* $\mathbb{R}^{\mathfrak{D}}$.

Remark. Now suppose that $\mathbf{M}$ is differentiable. Alas, the constructed embedding need not be differentiable.

However, the map Φ can be precomposed with a self-map of $\mathbb{R}^{\mathfrak{D}} \sqcup \{\infty\}$ which pulls the copy of $\mathbb{R}^{\mathfrak{D}}$ towards the point "∞" rapidly; sufficiently rapidly that each map $\widehat{h}_P$ has derivative zero at the boundary of P. □

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